FORCING WITH BUSHY TREES

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ABSTRACT. We present several results that rely on arguments involving the combinatorics of "bushy trees". These include the fact that there are arbitrarily slow-growing diagonally noncomputable (DNC) functions that compute no Kurtz random real, as well as an extension of a result of Kumabe in which we establish that there are DNC functions relative to arbitrary oracles that are of minimal Turing degree. Along the way, we survey some of the existing instances of bushy tree arguments in the literature.

1. INTRODUCTION

In 1985, Sacks [15] asked if there exist diagonally noncomputable (or DNC) functions of minimal Turing degree. Kumabe answered the question in 1993, constructing such a function and pioneering the application of bushy tree arguments in computability theory. A draft of the proof [12] was in private circulation by 1996, but has remained unpublished.

Arguments involving bushy trees and their combinatorics have since been applied to several questions concerning DNC functions. In 2000, Simpson and Giusto [7] asked if the reverse mathematics axiom system DNC is stronger than the system WWKL$_0$. In 2004, Ambos-Spies, Kjos-Hanssen, Lempp, and Slaman [1] used ideas from Kumabe's proof to provide an affirmative answer.

In 2009, motivated by questions around Yates's long-standing open problem about whether every minimal degree has a strong minimal cover, Lewis collaborated with Kumabe to produce a simplified version [13] of Kumabe's proof, the publication of which introduced the technique of "bushy tree forcing" to the wider community.

A simpler variation on the technique appeared in Greenberg and Miller's 2011 result [8] that there are arbitrarily slow-growing DNC functions that compute no Martin-Löf random real.

More recently, Beros [2] has applied arguments involving bushy trees to show that there exist DNC functions that compute no effectively bi-immune set, answering a question of Jockusch and Lewis [10]. Meanwhile, Dorais, Hirst, and Shafer [4], building on the aforementioned work of Ambos-Spies, et al. [1], have shown that the reverse mathematics principle "there exists a k such that for every function f there is a k-bounded function that is DNC relative f" does not imply the existence of a $\{0,1\}$-valued DNC function in the absence of Σ_2^0 induction, answering a question of Simpson.

We add two new results to this growing collection. Theorem 4.2 is a variation on the Greenberg-Miller result mentioned above, stronger in one aspect, but (necessarily) weaker in another: There are arbitrarily slow-growing DNC functions that

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compute no Kurtz random real, although this fact cannot be partially relativized to yield a DNC function relative to an arbitrary oracle. Theorem 5.1, due to the first author, is a partial relativization of Kumabe's theorem. It asserts the existence of DNC functions relative to arbitrary oracles that are of minimal Turing degree.

One of the goals of the current paper is to study the diverse applications of bushy tree arguments in computability theory with a view to understanding what the similarities and differences between them are. In the case of forcing arguments, we are particularly interested in how properties of the partial order determine properties of the generic object (typically a DNC function). The definitions and combinatorial lemmas in Section 2 underly all of the arguments we present, and encapsulate some of the similarities.

The differences can be seen to occur primarily along three "axes". The first of these relates to the nature of the approximation to the generic object. In some arguments, the approximations are finite strings (what we term "basic bushy forcing"), while others involve maintaining infinite trees. A second major difference is in the complexity of what we label the "bad set". These are sets of strings that are declared to be off limits in a construction. Some arguments (Theorem 4.2, for example) require that the bad sets be computably enumerable, and these are not automatically amenable to partial relativization. In others, dropping the assumption of any form of effectivity on the bad set allows partial relativization (as in Theorem 5.1), but may require more complicated combinatorics, or a different assumption on the effectivity of the approximation. The third major difference is in whether the resulting DNC function can be constructed pointwise below a preimposed order function. This is possible, for example, in Theorem 4.2, but the question of whether it is possible in Theorem 5.1 is an important open one.

2. Definitions and combinatorial lemmas

Definition 2.1. Given $\sigma \in \omega^{<\omega}$, we say that a tree $T \subseteq \omega^{<\omega}$ is *n-bushy above* σ if every element of T is comparable with σ, and for every $\tau \in T$ that extends σ and is not a leaf of T, τ has at least n immediate extensions in T. We will refer to σ as the *stem* of T.

Note that the set of initial segments of σ is actually n-bushy above σ according to the definition above.

Definition 2.2. Given $\sigma \in \omega^{<\omega}$, we say that a set $B \subseteq \omega^{<\omega}$ is *n-big above* σ if there is a finite n-bushy tree T above σ such that all its leaves are in B. If B is not n-big above σ then we say that B is *n-small above* σ.

We begin by establishing some of the basic combinatorial properties of bushy trees. The first observation is that we can extend the leaves of an n-bushy tree with n-bushy trees to obtain another n-bushy tree:

Lemma 2.3 (Concatenation property)**.** *Suppose that $A \subseteq \omega^{<\omega}$ is n-big above σ. If $A_\tau \subseteq \omega^{<\omega}$ is n-big above τ for every $\tau \in A$, then $\bigcup_{\tau \in T} A_\tau$ is n-big above σ.*

The second property that we use frequently is known as the *smallness preservation property*. This is the *second sparse subset property* of Kumabe and Lewis [13], and Lemma 5.4 of Greenberg and Miller [8].

Lemma 2.4 (Smallness preservation property). *Suppose that B and C are subsets of $\omega^{<\omega}$, that $m, n \in \omega$ and that $\sigma \in \omega^{<\omega}$. If B and C are respectively m-small and n-small above σ then $B \cup C$ is $(n + m - 1)$-small above σ.*

Proof. Let T be an $(m+n-1)$-bushy tree above σ with leaves in $B \cup C$. We show that either B is m-big above or C is n-big above σ. Label a leaf τ of T "B" if it is in B, "C" otherwise. Now if ρ is the immediate predecessor of τ, then ρ has at least $(m + n - 1)$ immediate extensions on T, each of which are labeled either "B" or "C". Then either m of these are labeled "B", in which case we label ρ "B", or n are labeled "C", in which case we label ρ "C". Continuing this process leads to σ eventually getting a label. It is clear that if σ is labeled "B" then B is m-big above σ. Otherwise C is n-big above σ. $\square$

The third property is known as the *small set closure property*:

Lemma 2.5 (Small set closure property). *Suppose that $B \subset \omega^{<\omega}$ is k-small above σ. Let $C = \{\tau \in \omega^{<\omega} : B \text{ is } k\text{-big above } \tau\}$. Then C is k-small above σ. Moreover C is k-closed, meaning that if C is k-big above a string ρ, then $\rho \in C$.*

Proof. Suppose that C is k-big above a string ρ. Then, since B is k-big above every $\tau \in C$, by the concatenation property, B is k-big above ρ, so $\rho \in C$. The lemma follows immediately. $\square$

The small set closure property is quite useful in the context of a forcing construction. Typically, σ is an approximation to a function that we are building and B is a set of strings that must be avoided in order to ensure that requirements remain met. We refer informally to the set B as the "bad set". Throughout the construction, we may wish to maintain the property that the bad set B is k-small above σ for some $k \in \omega$. Clearly, if B is k-big above some string ρ, then ρ is off-limits as well. Lemma 2.5 allows us to assume that all such strings are already in the bad set, while preserving its smallness. From now on, whenever we deal with a bad set that is k-small, we also assume that it is k-closed. Note that the k-closure of a c.e. set of strings is also c.e.

3. Basic bushy forcing

As a first illustration of the convenience afforded us by these lemmas, we present a proof of a well-known result. Any bounded DNC function (i.e., a function in DNC_k for some $k \geq 2$) computes a function in DNC_2. However, Jockusch showed in [9] that this is not uniform.

Theorem 3.1 (Jockusch [9]). *For each $n \geq 2$, there is no single functional Γ such that for all $f \in \text{DNC}_{n+1}$, $\Gamma^f \in \text{DNC}_n$.*

Proof. Let us assume that such a Γ exists, i.e., for all $f \in \text{DNC}_{n+1}$, $\Gamma^f \in \text{DNC}_n$. The set of sequences in DNC_{n+1} is a Π_1^0 class in $(n + 1)^\omega$, so we may obtain a functional Ξ that is total on $(n + 1)^\omega$ and that agrees with Γ on every member of DNC_{n+1}. We may also assume that $\Xi^f \in n^\omega$ for all $f \in (n + 1)^\omega$.

For each $m \in \omega$ and for each $i < n$, let $\Lambda_{i,m} = \{\sigma \in (n + 1)^{<\omega} : \Xi^\sigma(m) = i\}$. By the compactness of $(n + 1)^\omega$, there exists a finite level k such that for every string $\tau \in (n+1)^k$, $\Xi^\tau(m)$ converges. Therefore, $\bigcup_{i<n} \Lambda_{i,m}$ is $(n+1)$-big above the empty string $\langle\rangle$. It is now easy to see, by repeatedly applying the smallness preservation property, that for some $i < n$, $\Lambda_{i,m}$ must be 2-bushy above $\langle\rangle$.

We specify a partial computable function φ. On input m, φ searches for a 2-bushy tree T above $\langle\rangle$ such that for every leaf τ of T, $\Xi^\tau(m)$ converges to the same value, say i, and when it finds such a tree, itself outputs i. By the argument above, such a tree must exist, and so $\varphi(m)$ is defined for each m. Let e be the index for φ, and let T_e be the 2-bushy tree that φ finds on input e. Since T_e is 2-bushy, there is a leaf τ of T_e that is DNC, and so there is an $f \in (n+1)^\omega$ extending σ that is DNC_{n+1}. But then $\Xi^f(e) = \Xi^\tau(e) = \varphi_e(e)$, which is a contradiction. $\qquad\square$

Finitely iterating this strategy yields the following stronger result:

Theorem 3.2. *For each $n \geq 2$, there is no finite set of functionals $\Gamma_0, \Gamma_1, ..., \Gamma_k$ such that for all $f \in \mathrm{DNC}_{n+1}$, there exists a $j \leq k$ such that $\Gamma_j^f \in \mathrm{DNC}_n$.*

Proof. Let us assume that such a set of functionals exists. We define a new functional Ξ as follows: on input e, Ξ simulates Γ_0 through Γ_k on input e and outputs the result of whichever one converges first. Clearly, Ξ is total on the class of DNC_{n+1} sequences, so we can assume that Ξ is total on $(n+1)^\omega$. We then proceed exactly as in the proof of Theorem 3.1, obtaining a string σ_0 that is DNC_{n+1} and an $e \in \omega$ such that $\Xi^{\sigma_0}(e) = \varphi_e(e)$. Then $\Xi^{\sigma_0}(e) = \Gamma_j^{\sigma_0}(e)$ for some $j \leq k$. It follows that Γ_j fails to compute a DNC_n function on any $f \in \mathrm{DNC}_{n+1}$ extending σ_0. We now repeat the same process above σ_0 with the reduced list of functionals $\{\Gamma_1, ..., \Gamma_k\} \setminus \{\Gamma_j\}$, obtaining a DNC_{n+1} string σ_1 extending σ_0 that diagonalizes against one of the remaining functionals. After $k+1$ iterations, we will have obtained a contradiction. $\qquad\square$

The previous proof points the way towards more sophisticated constructions involving bushy trees where we satisfy countably many requirements. The next result is our first example of such a construction. It features a simpler variant of bushy tree forcing, which we term *basic bushy forcing*. In this type of forcing, the approximation to the generic object is a finite string.

Theorem 3.3 (Ambos-Spies, Kjos-Hanssen, Lempp, and Slaman [1])**.** *There is a* DNC *function that computes no computably bounded* DNC *function.*

Proof. The forcing conditions are pairs (σ, B), where $\sigma \in \omega^{<\omega}$, $B \subset \omega^{<\omega}$ and:

- for some $k \in \omega$, B is k-small above σ (and without loss of generality, k-closed)
- B is *upward closed* (i.e., if γ is in B, then all extensions of γ are in B).

The string σ is an approximation to f and the set B is a "bad set", i.e., a set of strings that must be avoided in order to ensure that requirements remain satisfied.

A condition (σ, B) *extends* another condition (τ, C) if $\tau \preceq \sigma$ and $C \subseteq B$. Let $\mathbb{P}$ denote this partial order. Now if $\mathcal{G}$ is a filter on $\mathbb{P}$, then for any two elements (σ, B) and (τ, C) of $\mathcal{G}$, σ and τ are comparable. Hence, $f_\mathcal{G} = \bigcup\{\sigma : (\sigma, B) \in \mathcal{G}\} \in \omega^{\leq\omega}$. In fact, we can ensure that $f_\mathcal{G}$ is total:

Claim 3.4. *If $\mathcal{G}$ is sufficiently generic with respect to $\mathbb{P}$, then $f_\mathcal{G}$ is total.*

Proof. We show that the collection $\mathcal{T}_m = \{(\sigma, B) \in \mathbb{P} : |\sigma| \geq m\}$ is dense in $\mathbb{P}$. Suppose $(\sigma, B) \in \mathbb{P}$, where $|\sigma| < m$. Then B is k-small above σ for some $k \in \omega$. Clearly, the set $C = \{\tau \in \omega^{<\omega} : |\tau| \geq m\}$ is k-big above σ. Let τ be any string in $C \setminus B$. Then $(\tau, B) \in \mathbb{P}$. $\qquad\square$

Claim 3.5. *If $\mathcal{G}$ is any filter on $\mathbb{P}$, then for all $(\sigma, B) \in \mathcal{G}$, $f_{\mathcal{G}}$ has no initial segment in B.*

Proof. Suppose that $f_{\mathcal{G}}$ has an initial segment τ in B. Then there is a $(\rho', C') \in \mathcal{G}$ such that ρ' extends τ. Let (ρ, C) be a common extension of (ρ', C') and (σ, B). Since B is upward closed, $\rho \in B$. But $B \subseteq C$, so $\rho \in C$. This is a contradiction, since it follows that C is k-big above ρ for all $k \in \omega$. $\qquad\square$

If Γ is a functional and h a computable function such that Γ is h-valued (in other words, whenever Γ converges with any oracle on input e, its output is less than $h(e)$), let $\mathcal{D}_{\Gamma,h}$ denote the set of $(\sigma, B) \in \mathbb{P}$ such that for all $g \in [\sigma] \setminus [B]^{\prec}$, Γ^g is not a DNC_h function.

Claim 3.6. *For each computable function h, and h-valued functional Γ, $\mathcal{D}_{\Gamma,h}$ is dense in $\mathbb{P}$.*

Proof. Suppose $(\sigma, B) \in \mathbb{P}$ and that B is k-small above σ. As in the proof of Theorem 3.1, we specify a partial computable function φ. On input m, φ searches for a k-bushy tree T above σ such that for every leaf τ of T, $\Gamma^\tau(m)$ converges to the same value $i < h(m)$. Upon finding such a tree, φ outputs i. Let e be the index of φ.

There are now two cases. If the set $A = \{\tau : \Gamma^\tau(e) \downarrow\}$ is $(h(e) \cdot k)$-small above σ, then $A \cup B$ is $(h(e) \cdot k + k - 1)$-small above σ. Then $(\sigma, A \cup B) \in \mathbb{P}$ and extends (σ, B). Note that we have forced Γ to be partial on any $g \in [\sigma] \setminus [A \cup B]^{\prec}$. Hence, $(\sigma, A \cup B) \in \mathcal{D}_{\Gamma,h}$.

On the other hand, if A is $(h(e) \cdot k)$-big above σ, then for some $i < h(e)$, $\{\tau : \Gamma^\tau(e) \downarrow = i\}$ is k-big above σ. So $\varphi(e)$ is defined. In this case, we extend σ to any τ not in B such that $\Gamma^\tau(e) \downarrow = \varphi(e)$. This forces Γ^g to fail to be DNC on any g extending τ. Hence, $(\tau, B) \in \mathcal{D}_{\Gamma,h}$. $\qquad\square$

Finally, we observe that the set of finite strings that are not DNC, which we denote by B_{DNC}, cannot be 2-big above the empty string $\langle\rangle$, since any 2-bushy tree contains a string that is DNC. So $(\langle\rangle, B_{\mathrm{DNC}}) \in \mathbb{P}$. Let $\mathcal{G}$ be a filter on $\mathbb{P}$ containing $(\langle\rangle, B_{\mathrm{DNC}})$ that meets $\mathcal{T}_m$ for every $m \in \omega$ and $\mathcal{D}_{\Gamma,h}$ for every computable function h and h-valued functional Γ (note that this is a countable collection of dense sets).

By Claim 3.4, $f_{\mathcal{G}}$ is total. By Claim 3.5 and the fact that $(\langle\rangle, B_{\mathrm{DNC}}) \in \mathcal{G}$, $f_{\mathcal{G}}$ is a DNC function. If $f_{\mathcal{G}}$ computes a function in DNC_h for some computable function h, then it does so via an h-valued functional Γ. Claim 3.6 shows that this is not the case. This concludes the proof of Theorem 3.3. $\qquad\square$

We note that while the bad sets in the previous proof are c.e., we do not make use of this fact. Given an oracle X, let B_{DNC}^X denote the set of finite strings that are not DNC *relative to* X. Note that B_{DNC}^X is not necessarily c.e., but is nevertheless 2-small above $\langle\rangle$. This suggests that we could use the same sort of techniques to construct a function that is DNC relative to X. As an example, we prove a theorem that implies the main result in [1], and is slightly stronger.

Theorem 3.7. *Fix an order function h. Suppose X computes no DNC_h function. Then there is an f that is DNC relative to X such that $f \oplus X$ computes no DNC_h function.*

Proof. The forcing partial order is the same as before. If Γ is an h-valued functional, let $\mathcal{D}_{\Gamma}$ denote the set of $(\sigma, B) \in \mathbb{P}$ such that for all $f \in [\sigma] \setminus [B]^{\prec}$, $\Gamma^{f \oplus X}$ is not a

DNC_h function. We show that $\mathcal{D}_\Gamma$ is dense in the partial order. Suppose (σ, B) is a condition where B is k-small above σ.

First, if there are $x, l \in \omega$ such that

$$C_x = \{\tau \in \omega^{<\omega} : \Gamma^{\tau \oplus X}(x)\downarrow\}$$

is l-small above σ, then the condition $(\sigma, B \cup C_x)$ extends (σ, B) and forces the divergence of $\Gamma^{f_\mathcal{G} \oplus X}(x)$. Therefore, let us assume that for each $x, l \in \omega$, C_x is l-big above σ.

Next, if there exists an $x \in \omega$ such that $\varphi_x(x)$ converges and

$$N_x = \{\tau \in \omega^{<\omega} : \Gamma^{\tau \oplus X}(x)\downarrow = \varphi_x(x)\}$$

is k-big above σ, then there is a τ extending σ not in B such that $\Gamma^{\tau \oplus X}(x)\downarrow = \varphi_x(x)$, and so the condition (τ, B) extends (σ, B) and forces that $f_\mathcal{G}$ is not DNC. Therefore, let us assume that for each $x \in \omega$, either $\varphi_x(x)$ diverges or N_x is k-small above σ.

We now describe how to compute a DNC_h function from X, which yields a contradiction. On input x, search for a k-bushy tree T above σ such that for every leaf τ of T, $\Gamma^{\tau \oplus X}(x)$ converges to the same value $j < h(x)$, then output j. Since for each x, C_x is $(h(x) \cdot k)$-big above σ, such a tree T exists. So the X-computable function just described is total. Moreover, it disagrees with $\varphi_x(x)$ whenever it is defined, since N_x is k-small above σ.

Therefore, $\mathcal{D}_\Gamma$ is dense. Let $\mathcal{G}$ be a generic filter including the condition $(\langle\rangle, B_{\mathrm{DNC}}^X)$. Then $f_\mathcal{G}$ has the required properties. $\qquad\square$

With a stronger assumption, the technique in the proof of Theorem 3.7 yields a stronger conclusion: If X computes no computably bounded DNC function, then there is an f that is DNC relative to X such that $f \oplus X$ computes no computably bounded DNC function. We omit the proof.

An analysis of the amount of bushiness we require above σ in the diagonalization argument of Claim 3.6 yields the following:

Theorem 3.8 (Ambos-Spies, et al. [1])**.** *For each order function h there is an order function j and a function $f \in \mathrm{DNC}_j$ that computes no function in DNC_h.*

Proof. If j is an order function, let j^n denote the space

$$\prod_{m<n} \{0, 1, ..., j(m) - 1\},$$

and let $j^{<\omega}$ and j^ω be defined in the obvious way.

We now fix a computable function h and let $(\Gamma_i)_{i\in\omega}$ be an effective enumeration of all h-valued Turing functionals. We define an order function j by recursion. In order to define j, we will also define an auxiliary computable function $q : \omega^{<\omega} \times \omega^2$, the definition of which will refer to the index of the function j. This is possible because we can assume, by the recursion theorem, that we have access to the index of j in advance.

On input x, $\varphi_{q(\sigma,i)}$ searches for a $|\sigma|$-bushy tree T above σ contained in $j^{<\omega}$ such that for every leaf τ of T, $\Gamma_i^\tau(x)$ converges to the same value $k < h(x)$, and upon finding such a tree, itself outputs k. Now let $\bar{q} = \max_{i<n, \sigma \in j^n} q(\sigma, i)$. We define $j(n)$ to be the larger of $\max_{i<n} j(i)$ and $((h(\bar{q}(n)) + 1) \cdot n) + 2$.

The forcing conditions are now pairs (σ, B) where $B \subseteq j^{<\omega}$ and $\sigma \in j^{<\omega} \setminus B$. We require that B be upward-closed and $|\sigma|$-small above σ. By the small set

closure property, we may assume that B is $|\sigma|$-closed. For $\sigma \in j^{<\omega}$, let $[\sigma]_j$ denote $\{X \in j^\omega : \sigma \prec X\}$.

Claim 3.9. *Let $\mathcal{D}_i$ denote the set of $(\sigma, B) \in \mathbb{P}$ such that for all $g \in [\sigma]_j \setminus [B]^{\prec}$, Γ_i^g is not a DNC_h function. Then for each $i \in \omega$, $\mathcal{D}_i$ is dense in $\mathbb{P}$.*

Proof. Suppose that $(\sigma, B) \in \mathbb{P}$. By suitably extending σ, we can assume that $|\sigma| > i$. Let $n = |\sigma|$ and

$$A = \{\tau \in j^{<\omega} : \Gamma_i^\tau(q(\sigma, i)) \downarrow\}.$$

As in the proof of Claim 3.6, there are two cases.

If A is $(h(q(\sigma, i)) \cdot n)$-small above σ, then letting $c = (h(q(\sigma, i)) \cdot n + n - 1)$, $A \cup B$ is c-small above σ. Let C be the c-closure of $A \cup B$. Since $j(n) \geq (h(q(\sigma, i)) + 1) \cdot n > c$ and j is nondecreasing, j^c is c-big above σ. Let τ be any string extending σ in $j^c \setminus C$. Then (τ, C) is a condition. Further, Γ_i^f is partial on any $f \in [\tau]_j \setminus [C]^{\prec}$, so $(\tau, C) \in \mathcal{D}_i$.

On the other hand, if A is $(h(q(\sigma, i)) \cdot n)$-big above σ, then for some $k < h(q(\sigma, i))$, the set $\{\tau \in j^{<\omega} : \Gamma_i^\tau(q(\sigma, i)) \downarrow = k\}$ is n-big above σ. It follows that $\varphi_{q(\sigma,i)}(q(\sigma, i))$ is defined. So there is a $\tau \in j^{<\omega} \setminus B$ extending σ such that $\Gamma_i^\tau(q(\sigma, i))) = \varphi_{q(\sigma,i)}(q(\sigma, i))$. Then $(\tau, B) \in \mathbb{P} \cap \mathcal{D}_i$. □

This concludes the proof of Theorem 3.8. □

Theorem 3.10. *Given any order function g, there is an order function h and an $f \in \mathrm{DNC}_g$ such that f computes no DNC_h function.*

Proof. We define h inductively. Let $n_0 = 0$ and let $h(0) = 2$. At the ith stage of the construction, suppose we have defined it up to n_i. Let $k \geq n_i + 1$ be the least such that $g(k) \geq (h(n_i) + 1) \cdot g(n_i)$. Let $q(\sigma)$ be the computable function such that if $\sigma \in g^k$, then $q(\sigma) \geq k$, and $\varphi_{q(\sigma)}(n)$ searches for a $g(n_i)$-bushy tree T above σ contained in $g^{<\omega}$ such that for every leaf τ of T, Φ_{i-1}^τ converges to the same value $l < h(n_i)$. Let $m = \max_{\sigma \in g^k} q(\sigma)$. Let $h(n) = h(n_i)$ for all n such that $n_i < n \leq m$ and let $h(m + 1) = h(m) + 1$. Finally, let $n_{i+1} = m + 1$, ensuring that h is unbounded. The fact that $k \geq n_i + 1$ ensures that h is total.

It remains to construct f. Let $B_0 = B_{\mathrm{DNC}}$ and let $\sigma_0 \in g^1 \setminus B_{\mathrm{DNC}}$. Assume inductively that $\sigma_i \in g^{n_i} \setminus B_i$ and that B_i is $g(n_i)$-small above σ_i. Let k and q be defined as above and extend σ to a string $\rho \in g^k \setminus B_i$. For $j < h(q(\rho))$, let

$$A_j = \{\tau \in g^{<\omega} : \Phi_i^\tau(q(\rho)) \downarrow = j\}.$$

If A_j is $g(n_i)$-big above ρ for some j, then $\varphi_{q(\rho)}(q(\rho))$ is defined. If $\varphi_{q(\rho)}(q(\rho)) = j'$ then there is a $\tau \in A_{j'} \setminus B_i$ extending ρ such that $\Phi_{i-1}^\tau(q(\rho)) = \varphi_{q(\rho)}(q(\rho))$. Otherwise, $C = (\bigcup_{j < h(q(\rho))} A_j) \cup B_i$ is $(h(q(\rho)) + 1) \cdot g(n_i)$-small above ρ. Since $g(k) \geq (h(n_i) + 1) \cdot g(n_i) = (h(q(\rho)) + 1) \cdot g(n_i)$, C is $g(k)$-small above ρ. So let $B_{i+1} = C$ and let σ_{i+1} be any string in $g^{n_{i+1}} \setminus B_{i+1}$ extending ρ. Finally, let $f = \bigcup_{i \in \omega} \sigma_i$. □

By alternating the strategies of Theorems 3.8 and 3.10, one can also show:

Theorem 3.11. *Given any order function g_0, there is another order function g_1 and functions $f_0 \in \mathrm{DNC}_{g_0}$ and $f_1 \in \mathrm{DNC}_{g_1}$ such that f_0 computes no DNC_{g_1} function and f_1 computes no DNC_{g_0} function.*

4. Bushy tree forcing

Bounded DNC functions, being of PA degree, compute Martin-Löf random reals. Kučera [11] showed that there is an order function h such that every Martin-Löf random real computes a DNC_h function. Theorem 3.3 then implies that there are unbounded DNC functions that compute no Martin-Löf random real. Greenberg and Miller established a stronger version of this fact:

Theorem 4.1 (Greenberg and Miller [8]). *For each order function h, there is an $f \in \mathrm{DNC}_h$ that computes no Martin-Löf random real.*

The proof uses basic bushy forcing, and does not require that the bad sets be c.e. In fact, the same technique could be used to show that for each order function h and each oracle X, there is an $f \in \mathrm{DNC}_h^X$ that computes no Martin-Löf random real. Our main result in this section cannot be partially relativized in this manner (it strongly depends on the fact that the bad sets are c.e.) but improves upon the Greenberg-Miller theorem in a different way. Recall that a real is *Kurtz random* (sometimes also called *weakly random*) if it is not contained in any measure 0 Π_1^0 class.

Theorem 4.2. *For each order function h, there is an $f \in \mathrm{DNC}_h$ that computes no Kurtz random real.*

Theorem 4.2 is our first example of *bushy tree forcing*, where the conditions consist of trees, not just finite strings. The atomic step in the forcing is based on the following result of Downey, Greenberg, Jockusch, Milans [5], which we prove here for convenience.

Theorem 4.3 (Downey, et al. [5]). *There is no single functional Γ such that Γ^f is Kurtz random for all $f \in \mathrm{DNC}_3$.*

Proof. Suppose that such a functional Γ exists. As before, we may assume that Γ is total. It will be convenient to assume that Γ satisfies the following additional property:

- If $\sigma \in 3^{<\omega}$ and $\Gamma^\sigma(n)$ converges, then $\Gamma^\sigma(n)$ converges within $|\sigma|$ steps and for all $n' < n$, $\Gamma^\sigma(n')$ also converges.

It is not difficult to see that this assumption can be made without any loss of generality and that if Γ satisfies this property, then $\Gamma^\sigma = \tau$ is a computable relation for $\sigma \in 3^{<\omega}$ and $\tau \in 2^{<\omega}$.

We build a computable 2-bushy subtree S of 3^ω with no leaves such that the image of Γ on S (denoted by $\Gamma(S)$) has measure 0. The tree S will be obtained as the union of a sequence $\{\langle\rangle\} = S_0 \subset S_1 \subset S_2...$ of finite regular[1] binary subtrees of $3^{<\omega}$. Let $\Gamma(S_i)$ denote the set of reals

$$\bigcup\{[\Gamma^\sigma] : \sigma \text{ is a leaf of } S_i\}.$$

In constructing S_{i+1}, we want to ensure that $\mu(\Gamma(S_{i+1})) \leq (3/4)\mu(\Gamma(S_i))$. Let $L = \{\sigma_0, \sigma_1, ..., \sigma_{|L|-1}\}$ be the set of leaves of S_i and let $m = \max\{|\Gamma^\sigma| : \sigma \in L\}$. Our assumption on Γ above allows us to find m computably. Let l be large enough so that for all $\tau \in 3^l$, $|\Gamma^\tau| \geq m + (2^{|L|} + 1)$. In other words, l is large enough so that we obtain at least $2^{|L|} + 1$ *additional* bits of convergence by extending a leaf of

---

[1] All the leaves are of the same length.

S_i to any ternary string of length l. Note that such an l exists by the compactness of 3^ω and that we can find it computably. Let $T_j = \{\tau \in 3^l : \tau \succ \sigma_j\}$.

Suppose that k is a position corresponding to one of the additional bits of convergence, i.e., $m \le k < m + 2^{|L|} + 1$. Since each T_j is 3-big above σ_j, by the smallness preservation property, either $\{\tau \in T_j : \Gamma^\tau(k) = 1\}$ is 2-big above σ_j (in which case, we say that we can *force* the kth bit to be 1 above σ_j) or $\{\tau \in T_j : \Gamma^\tau(k) = 0\}$ is 2-big above σ_j (we say that we can force the kth bit to be 0 above σ_j). This allows us to obtain a binary sequence ρ_k of length $|L|$, where $\rho_k(j) = 1$ if we can force the kth bit to be 1 above σ_j, and 0 otherwise. Moreover, we can computably find 2-big sets above σ_j that force the kth bit one way or another, so we can compute ρ_k, given k.

By the pigeonhole principle, there exist r and s such that $m \le r, s < m + 2^{|L|} + 1$ and $\rho_r = \rho_s$. Note that for each $j < |L|$, even though we can force the rth and s^{th} bits in the same way above σ_j, we may not be able to do so *simultaneously*. We adopt the following strategy above each σ_j: If we can force the rth bit to be 1 above σ_j, we do so, by extending σ_j to a finite 2-bushy tree B_j with leaves in 3^l such that for every leaf τ of B_j, $\Gamma^\tau(r) = 1$. Otherwise, $\rho_r(j) = \rho_s(j) = 0$, so we force the sth bit to be 0 above σ_j. The regular binary tree of height l that results is S_{i+1}.

For any leaf τ of S_{i+1}, it is *not* the case that the rth bit of Γ^τ is 0 and the sth bit is 1: Say τ extends σ_j. By our choice of strategy, if the rth bit is 0, then it must be the case that we could not have forced it to be 1 above σ_j, and so we would have forced the sth bit to be 0 above σ_j.

Let $P = \{X \in \Gamma(S_i) : X(r) = 0 \text{ and } X(s) = 1\}$. Then $\mu(P) = (1/4)\mu(\Gamma(S_i))$, since $r, s \ge m$. Clearly, $\Gamma(S_{i+1}) \subseteq \Gamma(S_i) \setminus P$, so $\mu(\Gamma(S_{i+1})) \le (3/4)\mu(\Gamma(S_i))$, as desired.

Let $S = \bigcup_{i \in \omega} S_i$. Then $\mu(\Gamma(S)) = \mu(\bigcap_{i \in \omega} \Gamma(S_i)) = 0$. Let f be any path through S that is DNC$_3$. Then $\Gamma^f \in \Gamma(S)$. But $\Gamma(S)$ is a null Π_1^0 class, which implies that Γ^f is not Kurtz random, contradicting our initial assumption. $\square$

Note that the construction in Theorem 4.3 starts with a 3-bushy tree and produces a 2-bushy subtree with no leaves.

Definition 4.4. Let j be an order function. We say that a tree $T \subseteq \omega^{<\omega}$ is *j-bushy* above a string $\sigma \in \omega^{<\omega}$ if every element of T is comparable with σ and for each τ extending σ that is not a leaf of T, there are at least $j(|\tau|)$ many immediate extensions of τ. We say T is *exactly j-bushy above σ* if for each nonleaf τ, there are exactly $j(|\tau|)$ immediate extensions of τ in T.

Proof of Theorem 4.2. The forcing conditions have the form (σ, T, B), where $\sigma \in \omega^{<\omega}$, T is a computable subtree of $\omega^{<\omega}$, $B \subset T$ and:

- T is exactly j-bushy above σ for some order function j
- B is c.e. and upward-closed in T (i.e., if $\tau \in B$ then ρ extending τ on T is also in B)
- B is $j(|\sigma|)$-small above σ (and, without loss of generality, $j(|\sigma|)$-closed).

A condition (σ, T, B) *extends* another condition (τ, S, C) if $\sigma \succeq \tau$, $T \subseteq S$ and $B \cap T \supseteq C \cap T$. Let $\mathbb{P}$ denote this partial order. As before, if $\mathcal{G}$ is a filter on $\mathbb{P}$, then $f_\mathcal{G} = \bigcup\{\sigma : (\sigma, T, B) \in \mathcal{G}\} \in \omega^{\le\omega}$. It is not difficult to verify that if $\mathcal{G}$ is sufficiently generic, then $f_\mathcal{G}$ is total and if $(\sigma, T, B) \in \mathcal{G}$, then $f_\mathcal{G}$ contains no initial segment in B.

If Γ is any functional, let $\mathcal{D}_\Gamma$ denote the set of $(\sigma, T, B) \in \mathbb{P}$ such that either

- $g \in [T] \setminus [B]^{\prec}$ implies that Γ^g is total, or
- there is an $n \in \omega$ such that $g \in [T] \setminus [B]^{\prec}$ implies that $\Gamma^g(n)\uparrow$.

Claim 4.5. $\mathcal{D}_\Gamma$ *is dense in* $\mathbb{P}$.

Proof. Suppose $(\sigma, T, B) \in \mathbb{P}$, where T is exactly j-bushy above σ. Let $C_x = \{\tau \in T : \Gamma^\tau(x)\downarrow\}$. Note that C_x is c.e. and upward closed in T. As usual, there are two cases.

Case 1. For every $\tau \in T$ extending σ and every $x \in \omega$, $C_x \cup B$ is $j(|\tau|)$-big above τ. In this case, we build a computable tree $S \subseteq T$ in stages that is exactly j'-bushy above σ for an order function j'. Let S_0 consist of just σ and its initial segments. Suppose inductively that we have $l_i \in \omega$ and $S_i \subset T$ such that

- For each $x < l_i$, $j'(x)$ has already been defined and $j'(x) \leq j(x)$.
- S_i is a finite, regular j'-bushy tree of height l_i above σ.
- For every leaf τ of S_i, either $\Gamma^\tau(x)\downarrow$ for every $x < i$ or $\tau \in B$.

Let τ be a leaf of S_i. By assumption, $C_i \cup B$ is $j(|\tau|)$-big above τ, so we extend τ to a finite tree with leaves in $C_i \cup B$ that is $j(|\tau|)$-bushy above τ. Note that since $C_i \cup B$ is c.e., we can find such a tree computably. The tree S'_{i+1} that results from carrying out this operation above each leaf of S_i may not be regular, but since both C_i and B are upward closed in T and T is j-bushy above the leaves of S'_{i+1}, we can extend them $j(l_i)$-bushily to some common level l_{i+1}, retaining the property that every leaf is in C_i or in B, and producing the tree S_{i+1}. We now let $j'(x) = j(l_i)$ for $l_i \leq x < l_{i+1}$. Note that j' is nondecreasing because of our assumption that $j'(x) \leq j(x)$ for $x < l_i$.

Let $S = \cup_{i \in \omega} S_i$ and note that since $j'(|\sigma|) = j(|\sigma|)$, B is already $j'(|\sigma|)$-closed. So the condition $(\sigma, S, B \cap S)$ extends (σ, T, B). Finally, if $g \in [S] \setminus [B]^{\prec}$, then for every i, $g \restriction l_i \in C_i$, so Γ^g is total.

Case 2. Let τ and x be counterexamples to the assumption in Case 1 and let S be the full subtree of T above τ. Let $B' = (C_x \cup B) \cap S$. Then B' is $j(|\tau|)$-small above τ, so $(\tau, S, B') \in \mathbb{P}$ and if $g \in [S] \setminus [B']^{\prec}$, then $\Gamma^g(x)$ diverges. $\qquad\square$

Let $\mathcal{H}_\Gamma$ be the set of all conditions (σ, T, B) such that if $g \in [T] \setminus [B]^{\prec}$, then Γ^g is not Kurtz random.

Claim 4.6. $\mathcal{H}_\Gamma$ *is dense in* $\mathbb{P}$.

Proof. Let $(\sigma, T, B) \in \mathbb{P}$ and Γ be a $\{0, 1\}$-valued functional. Claim 4.5 allows us to assume that Γ is total on $[T] \setminus [B]^{\prec}$, and since B is c.e., we can assume further that Γ is total on $[T]$. Let j be the order function such that T is exactly j-bushy above σ.

The remainder of the proof is a straightforward modification of Theorem 4.3. We build an order function j' and an exactly j'-bushy tree $S \subseteq T$ above σ in stages. Let S_0 consist of σ and its initial segments. Next, suppose inductively that we have $l_i \in \omega$ and $S_i \subset T$ such that

- For each $x < l_i$, $j'(x)$ has already been defined and $j'(x) \leq j(x)$.
- S_i is a finite, regular j'-bushy tree of height l_i above σ.

Let $\Gamma(S_i)$ denote $\{\Gamma^g : g \in [T] \cap [S_i]^{\prec}\}$.

We first extend S_i $j(l_i)$-bushily within T to a height $q > l_i$ such that $j(q) \geq 2j(l_i)$, obtaining the tree S'_{i+1}. This ensures that every level of T above q is $2j(l_i)$-big above each leaf of S'_{i+1}. Clearly, $\mu(\Gamma(S'_{i+1})) \leq \mu(\Gamma(S_i))$. Let L be the set of leaves of

S'_{i+1} and let $m = \max\{|\Gamma^\rho| : \rho \in L\}$. We choose l_{i+1} large enough so that for every $\tau \in T$ of length l_{i+1}, $|\Gamma^\tau| \geq m + 2^{|L|} + 1$. Note that the fact that T is exactly j-bushy ensures that we can find l_{i+1} computably.

For any leaf ρ of S'_{i+1}, let T_ρ be the set of strings of length l_{i+1} in T extending ρ. If k is a position corresponding to one of the additional bits of convergence (i.e., $m \leq k < m + 2^{|L|} + 1$), we say we can force the kth bit to be $c \in \{0, 1\}$ above ρ if $\{\tau \in T_\rho : \Gamma^\tau(k) = c\}$ is $j(l_i)$-big above ρ. Since T_ρ is $2j(l_i)$-big above ρ, if we cannot force the kth bit to be 0 above ρ, we can force it to be 1.

As in the proof of Theorem 4.3, we obtain positions r and s such that above each leaf of S'_{i+1}, the rth and s^{th} bits can be forced in the same way. We adopt the same strategy as before for extending S'_{i+1} to S_{i+1} and ensuring that $\mu(\Gamma(S_{i+1})) \leq (3/4)\mu(\Gamma(S_i))$. Finally, we let $j'(x) = j(l_i)$ for $l_i \leq x < l_{i+1}$.

Let $S = \bigcup_{i \in \omega} S_i$. Since $j'(|\sigma|) = j(|\sigma|)$, $B \cap S$ is $j'(|\sigma|)$-small above σ. So $(\sigma, S, B \cap S) \in \mathbb{P}$ and since $\mu(\Gamma(S)) = \mu(\bigcap_{i \in \omega} \Gamma(S_i)) = 0$, $(\sigma, S, B \cap S) \in \mathcal{H}_\Gamma$. $\qquad\square$

To conclude the proof of Theorem 4.2, let $\mathcal{G}$ be any filter containing $(\langle\rangle, h^{<\omega}, B_{\text{DNC}})$ that meets $\mathcal{H}_\Gamma$ for each functional Γ as well as the families of conditions that ensure totality. Then $f_\mathcal{G} \in \text{DNC}_h$ and does not compute a Kurtz random. $\qquad\square$

Every hyperimmune degree contains a Kurtz random [14], so if the function we are building is to avoid computing a Kurtz random, it must be hyperimmune-free. This is, in fact, the case:

Claim 4.7. *If $\mathcal{G}$ is sufficiently generic, then $f_\mathcal{G}$ has hyperimmune-free degree.*

Proof. Suppose $\Gamma^{f_\mathcal{G}}$ is a total function. Then if $(\sigma, T, B) \in \mathcal{G} \cap \mathcal{D}_\Gamma$, it must be the case that Γ is total on $[T] \setminus [B]^\prec$. Let Ξ be the functional that on input x and oracle $\tau \in T$, computes $\Gamma^\tau(x)$ until the computation converges or τ enters B. If the latter occurs first, then let $\Xi^\tau(x) = 0$. Now Ξ is total on $[T]$ and agrees with Γ on $[T] \setminus [B]^\prec$.

Let j be the order function such that T is exactly j-bushy above σ. We define a computable function m that majorizes $\Gamma^{f_\mathcal{G}}$. To compute $m(i)$, search for a finite tree $S_i \subset T$ that is j-bushy above σ such that for every leaf τ of S_i, $\Xi^\tau(i) \downarrow$. Note that such a finite tree must exist by the compactness of $[T]$ and we can find it computably since T is computable. Now let $m(i)$ be the maximum of the values $\Xi^\tau(i)$ as τ ranges over the leaves of S_i.

Since T is exactly j-bushy above σ and S_i is a subtree of T that is j-bushy above σ, $[T] \subseteq [S_i]^\prec$. So $f_\mathcal{G} \in [S_i]^\prec$ and $\Gamma^{f_\mathcal{G}}(i) = \Xi^{f_\mathcal{G}}(i) \leq m(i)$. $\qquad\square$

5. A DNC^X FUNCTION OF MINIMAL DEGREE

In this section, we strengthen Kumabe's result that there is a DNC function of minimal degree.

Theorem 5.1 (Khan)**.** *Given any oracle X, there is a function that is DNC relative to X and of minimal degree.*

Kumabe and Lewis [13] provided a simplified version of Kumabe's original arguments [12]. Our proof reuses much of the combinatorial machinery developed in the Kumabe-Lewis proof, but differs in several key aspects. Kumabe and Lewis use partial trees with computable domains, hence the function they produce is

hyperimmune-free. We use partial trees with noncomputable domains, out of ne-
cessity: by Theorem 6.1, any DNC function relative to $0'$ is hyperimmune. Further,
it suffices in the Kumabe-Lewis construction to work with bad sets of constant
bushiness. This is not the case here; our bad sets are h-small for some order func-
tion h. In our approach to bad sets of varying bushiness, we use ideas from Cai and
Greenberg's result in [3] that there exist degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$ is minimal
and DNC and $\mathbf{b}$ is DNC relative to $\mathbf{a}$ and a strong minimal cover of $\mathbf{a}$.

5.1. **Definitions and notation.**

Definition 5.2. Let h be an order function. Given $\sigma \in \omega^{<\omega}$, we say that a set
$B \subseteq \omega^{<\omega}$ is h-*big above* σ if there is a finite h-bushy tree T above σ such that all
its leaves are in B. If B is not h-big above σ then we say that B is h-*small* above
σ.

It is easy to see that the smallness preservation property, concatenation property
and small set closure property all continue to hold when one replaces the constants
governing bushiness with order functions.

For an order function g and $l \in \omega$, let $w(g,l)$ denote $\prod_{i<l} g(i)$ and let $r(g,l)$
denote $2^{3+3w_g(l)}$.

In order to simplify our calculations, throughout this proof we restrict ourselves
to order functions that only take values that are powers of two.

Definition 5.3. Suppose $h(n) = 2^{h'(n)}$ and $g(n) = 2^{g'(n)}$ are order functions,
where $h', g' : \omega \to \omega$. The *middle* of h and g is the order function $\mathcal{M}(h,g)$ defined
by
$$\mathcal{M}(h,g)(n) = 2^{\lfloor \frac{h'(n)+g'(n)}{2} \rfloor}.$$

Definition 5.4. Suppose h and g are order functions. We say the pair (h,g) *allows
splitting above* $N \in \omega$ if

   (1) $h(N) \geq g(N)$,
   (2) for $n \geq N$, $h(n)/g(n)$ is nondecreasing, and
   (3) there is an increasing sequence $\langle l_i \rangle_{i \in \omega}$ of natural numbers with $l_0 \geq N$ such
       that $h(l_i)/g(l_i) \geq (r(h,l_i))^i$.

We say (h,g) allows splitting if it allows splitting above some $N \in \omega$. We call the
sequence $\langle l_i \rangle$ the *splitting levels* for (h,g).

Lemma 5.5. *Let h and g be order functions such that (h,g) allows splitting. Let
$m = \mathcal{M}(h,g)$. Then (m,g) and (h,m) allow splitting.*

Proof. We provide the argument for (m,g). Suppose (h,g) allows splitting above
N and $\langle l_i \rangle$ is the sequence of splitting levels for (h,g). Note that conditions (1) and
(2) in the definition above are satisfied by (m,g) above N.

We verify condition (3). Suppose that $h(n) = 2^{h'(n)}$ and $g(n) = 2^{g'(n)}$. Note
first that for each $n \geq N$,
$$\frac{m(n)}{g(n)} = 2^{\lfloor \frac{h'(n)+g'(n)}{2} \rfloor - g'(n)} = 2^{\lfloor \frac{h'(n)-g'(n)}{2} \rfloor} \geq \frac{2^{\frac{h'(n)-g'(n)}{2}}}{2}.$$

It follows that for each $i \in \omega$,
$$\frac{m(l_{2i+2})}{g(l_{2i+2})} \geq \frac{2^{\frac{h'(l_{2i+2})-g'(l_{2i+2})}{2}}}{2} \geq \frac{(r(h,l_{2i+2}))^{\frac{2i+2}{2}}}{2} = \frac{(r(h,l_{2i+2}))^{i+1}}{2} \geq (r(m,l_{2i+2}))^i,$$

so $\langle l_{2i+2} \rangle_{i \in \omega}$ is a sequence of splitting levels for (m, g).

A similar calculation shows that (h, m) allows splitting. $\qquad \square$

It is not hard to verify that if (h, g) allows splitting then for any $c \in \omega$, so do $(h, 2^c g)$ and $(\max(h/2^c, 2), g)$.

5.2. **The partial order.** The forcing conditions are of the form (σ, T, B, h_T, h_B), where

- the tree T is partial recursive (some nodes may be terminal) and exactly h_T-bushy above σ
- B includes the terminal nodes in T, is upward closed and is h_B-small above σ
- (h_T, h_B) allows splitting above $|\sigma|$.

Only σ, T and B contribute to the ordering. Let h_M denote $\mathcal{M}(h_T, h_B)$. By extending σ appropriately, we can assume that $h_M(n)/16 \geq h_B(n)$ for all $n \geq |\sigma|$.

Note that we have no access to the set B (it is not c.e.). Since the terminal nodes of T are contained in the bad set B, the conditions that force $f_\mathcal{G}$ to be total are dense in this partial order.

As before, we can assume that the bad set is h_B-closed. In other words, if τ is any string in $T \setminus B$ then B is h_B-small above τ.

5.3. **Forcing $\Gamma^{f_\mathcal{G}}$ to be partial.** Let $C_n = \{\tau \in T : \Gamma^\tau(n)\!\downarrow\}$. Given a condition (σ, T, B, h_T, h_B) and a functional Γ we say we can *force $\Gamma^{f_\mathcal{G}}$ to be partial* if there is a τ on T extending σ and an n such that the set $C_n \cup B$ is h_M-small above τ. If this is the case, then we let T' be the full subtree of T above τ. The condition $(\tau, T', C_n \cup B, h_T, h_M)$ extends (σ, T, B, h_T, h_B), while forcing $\Gamma^{f_\mathcal{G}}(n)\!\uparrow$. From now on we assume that we cannot force $\Gamma^{f_\mathcal{G}}$ to be partial. It follows that for every n, and every $\tau \in T \setminus B$, $C_n \setminus B$ is $h_M/2$-big above τ. Applying this fact iteratively we obtain the following claim:

Claim 5.6. *For any $\tau \in T \setminus B$ extending σ and any n, there is an $A \subset T \setminus B$, $h_M/2$-big above τ, such that for every $\rho \in A$, $\Gamma^\rho \!\restriction\! n$ is defined.*

5.4. **Forcing $\Gamma^{f_\mathcal{G}}$ to be computable.** It is worth pointing out here how our argument for this case of the forcing differs from the one in Kumabe-Lewis. As we have mentioned, the bad sets in their argument are c.e., and they make strong use of this fact in an *effective* simultaneous construction of a refined subtree and a real such Y that it is the image of Γ on every path on this subtree (and hence computable). We do not have access to the bad set, since we will ultimately want it to include the set of strings that are non-DNC relative to X. So we construct Y *noneffectively*, arguing that it is the image under Γ of a sufficiently bushy subtree. Under the assumption that we make in this case of the forcing, Y will turn out to be computable.

Definition 5.7. Let g be an order function. A *g-big splitting above $\tau \in T$* is a pair of sets $A_0 \subset T$ and $A_1 \subset T$, both g-big above τ, such that for any $\tau_0 \in A_0$ and $\tau_1 \in A_1$, $\Gamma^{\tau_0} \mid \Gamma^{\tau_1}$. We say that A_0 and A_1 are Γ-*splitting*.

Suppose that there is a $\tau \in T \setminus B$ extending σ such that we cannot find any $h_M/16$-big splitting above τ. Under this assumption, we construct a real Y with the property that for each $n \in \omega$, the set of ρ on T such that $\Gamma^\rho \!\restriction\! n = Y \!\restriction\! n$ is $h_M/4$-big above τ. It follows immediately that Y is computable: to compute

it up to n bits, we search for an $h_M/4$-bushy tree $A \subset T$ above τ every leaf of which gives the same n bits of convergence via Γ. These bits must agree with Y, otherwise we will have obtained an $h_M/16$-big splitting above τ. Further, if we let $D = \{\rho \in T : \Gamma^\rho \mid Y\}$, then D is $h_M/16$-small above τ. It follows that $B \cup D$ is h_M-small above τ, so letting T' be the full tree above τ, the condition $(\tau, T', B \cup D, h_T, h_M)$ extends (σ, T, B, h_T, h_B) while forcing $\Gamma^{f_\mathcal{G}}$ to be computable.

We construct Y bit by bit (although not effectively), letting $Y_0 = \Gamma^\tau$. We also assume inductively that there is a set $S_i \subset T \setminus B$ that is $h_M/4$-big above τ and for every $\rho \in S_i$, $\Gamma^\rho \upharpoonright i + |Y_0| = Y_i$. Let S_0 consist of just τ.

Given Y_i and S_i, we proceed as follows. Above each leaf ρ of S_i, there is an $h_M/2$-big set of strings A_ρ such that for each $\nu \in A_\rho$, $\Gamma^\nu(|Y_i|)$ is defined. A_ρ can then be thinned out to a set A'_ρ that is $h_M/4$-big above ρ and such that for each $\nu \in A'_\rho$, $\Gamma^\nu(|Y_i|)$ converges to the same value c_ρ. Next, since S_i is $h_M/4$-big above τ, there is a $V \subset S_i$, $h_M/8$-big above τ, such that for each $\rho \in V$, c_ρ is the same value, say j. Let $Y_{i+1} = Y_i j$. Note that $V' = \cup\{A'_\rho : \rho \in V\}$ is $h_M/8$-big above τ and for each $\nu \in V'$, $\Gamma^\nu \succeq Y_{i+1}$. Let $S_{i+1} = \{\nu \in C : \Gamma^\nu \succeq Y_{i+1}\}$. The set $C \setminus S_{i+1}$ must be $h_M/16$-small above τ, otherwise $C \setminus S_{i+1}$ and V' form an $h_M/16$-big splitting above τ. It follows that S_{i+1} is $h_M/4$-big above τ.

5.5. Forcing $\Gamma^{f_\mathcal{G}} \geq_T f_\mathcal{G}$.
We work now under the additional assumption that for each $\tau \in T \setminus B$ extending σ there is a $h_M/16$-big splitting above τ.

We refine T to a subtree S that has the *delayed splitting property*: above each $\tau \in S \setminus B$, there are levels $l' > l > |\tau|$ such that if ρ_0 and ρ_1 are any two extensions of τ on S of length l, and $\rho'_0 \succ \rho_0$ and $\rho'_1 \succ \rho_1$ are extensions on S of length l', then $\Gamma^{\rho'_0} \mid \Gamma^{\rho'_1}$.

The statement of the following lemma has been slightly modified from the original in order to apply to trees of varying bushiness:

Lemma 5.8 (Kumabe, Lewis [13]). *Let Γ be a functional. Let A be $4g$-big above α and B be $4h$-big above β, where g and h are order functions. Suppose that above every leaf τ of A, there exist $\Delta_{\tau,0}$ and $\Delta_{\tau,1}$, such that they are both $4g$-big above τ and are Γ-splitting. Let $A' = \cup_{\tau,i}\Delta_{\tau,i}$ and let $v = \max\{|\Gamma^\rho| : \rho \in A'\}$. If for every leaf σ of B, $|\Gamma^\sigma| > v$, then there is an $A'' \subseteq A'$ and a $B' \subseteq B$, g-big above α and h-big above β respectively, that are Γ-splitting.*

Proof. Let $\sigma_0 = \langle\rangle$ and $B_0 = B$.

Assume inductively that we have σ_s of length s and B_s, h-big above β, such that for all $\rho \in B_s$, $\Gamma^\rho \succeq \sigma_s$.

If $\{\tau \in A' : \Gamma^\tau \mid \sigma_s\}$ is g-big above α then we are done. If not, then either

(1) $A_1 = \{\tau \in A' : \Gamma^\tau \preceq \sigma_s\}$ is g-big above α or
(2) $A_2 = \{\tau \in A' : \Gamma^\tau \text{ properly extends } \sigma_s\}$ is g-big above α.

If (1) holds then let V be the set of leaves of A that have an extension in A_1. For each $\tau \in V$, the set of strings in A_1 extending τ must lie entirely in one of the $\Delta_{\tau,i}$. Let Δ'_τ denote the other member of the splitting above τ. Then $\cup\{\Delta'_\tau : \tau \in V\}$ is g-big above α and splits with B_s.

Next, assume (2) holds, which implies that $|\sigma_s| < v$. If $\{\tau \in B : \Gamma^\tau \mid \sigma_s\}$ is h-big above β, then we are done. If not, then it must be the case that $D = \{\tau \in B : \Gamma^\tau \succeq \sigma_s\}$ is $h/2$-big above β. D can be partitioned into the sets $D_i = \{\tau \in D : \Gamma^\tau(|\sigma_s|) = i\}$, one of which must be $h/4$-big above β, say D_j. Let $B_{s+1} = D_j$ and let $\sigma_{s+1} = \sigma j$

and continue the construction. Since this process cannot continue indefinitely, we will obtain the required splitting via one of the other alternatives. □

Claim 5.9. *Suppose $\tau_0, ..., \tau_k$ are nodes of length l in $T \setminus B$, $k < w_{h_M}(l)$ and that $h_M(l)/h_B(l) \geq r(h_M, l)$. Then there is a sequence of sets $A_0, ..., A_k$, where A_j is $(h_M/2^{3+3k})$-big above τ_j and which are pairwise Γ-splitting.*

Proof. The proof is by induction on k. Suppose we already have $A_0, ..., A_k$, where each A_j is $(h_M/2^{3+3k})$-big above τ_j and the collection is pairwise Γ-splitting. Let τ_{k+1} be an additional node of length l that is not in B and let $q = h_M/2^3$.

Note that since $w_{h_M}(l) > k+1$, $h_B(l) < q(l)/2^{3k+1}$. So we first refine each A_j to a Π_j where Π_j is $(q/2^{3k+1})$-big above τ_j and $\Pi_j \cap B = \emptyset$. If ρ is a leaf of Π_j, then it is not in B and since $q/2^{3k+1} \leq h_M/16$, we can find a $q/2^{3k+1}$-bushy splitting, say $D_{\rho,0}$ and $D_{\rho,1}$, above ρ. We let $\Pi'_j = \cup_{i,\rho} D_{\rho,i}$.

Let m be the longest length of the image of Γ on any string in any of the Π'_j. Appealing to Claim 5.6, we let Δ_0 be a q-big set above τ_{k+1} such that each leaf of Δ_0 gives at least $m+1$ bits of convergence via Γ. We now apply Lemma 5.8 on Π'_0 and Δ_0, obtaining $A'_0 \subset \Pi'_0$ and $\Delta_1 \subset \Delta_0$, which are Γ-splitting and where the former is $q/2^{3(k+1)}$-big above τ_0 and the latter is $q/4$-big above τ_{k+1}. Next, we apply Lemma 5.8 to the pair Π'_1 and Δ_1, obtaining $A'_1 \subset \Pi'_1$ and $\Delta_2 \subset \Delta_1$, which are Γ-splitting and where A'_1 is $q/2^{3(k+1)}$-big above τ_1 and Δ_2 is $q/4^2$-big above τ_{k+1}. After $k+1$ applications of Lemma 5.8, we will have obtained A'_0 through A'_k and Δ_{k+1}, which are pairwise Γ-splitting. Moreover, Δ_{k+1} is $q/2^{2(k+1)}$-big above τ_{k+1}, so we can let $A'_{k+1} = \Delta_{k+1}$. □

Our argument here differs once again in a crucial way from Kumabe and Lewis's. Suppose we have defined the delayed splitting tree S up to a certain level and let τ be one of the leaves of this finite tree. In order to continue the construction above τ, we must find a sufficiently bushy splitting above τ. In the Kumabe-Lewis argument, such a splitting will be found, or τ will be seen to enter the bad set. In either case, the construction of the tree S is in no danger of "stalling". Here, however, we have no access to the bad set, so we may end up searching in vain for a splitting. In order to get around this, we will only ask for splittings above sufficiently bushy many leaves of the current approximation to S, a situation that we can guarantee, and add the remaining leaves to the bad set. Thus, we will be adding lots of strings to the bad set at each level of the construction. The following lemma is critical to preserving its smallness when we do so:

Lemma 5.10. *Let g be an order function. Suppose $A \subset \omega^{<\omega}$ is g-small above $\sigma \in \omega^{<\omega}$, and suppose $\tau \in \omega^{<\omega}$ extends σ and A contains no extension of τ. If B is a set of strings extending τ that is g-small above τ, then $A \cup B$ is g-small above σ.*

Proof. Suppose otherwise, i.e., there is a g-bushy tree T above σ with leaves in $A \cup B$. Clearly, some leaves of T are in B. Since every string in B extends τ, $\tau \in T$. This means that there is a tree T' that is g-bushy above τ whose leaves are in B, namely the tree consisting of all strings in T that are comparable with τ. This is a contradiction. □

Let $\langle l_i \rangle$ be the sequence of splitting levels for the pair (h_M, h_B). We begin by defining h_S. Let $j_i = l_{i+1}$. For $n < j_0$, let $h_S(n) = h_M(n)$. For $j_{i+1} > n \geq j_i$, let

$h_S(n) = h_M(j_i)/r(h_S, j_i)$. Then for each i,

$$\frac{h_S(j_i)}{h_B(j_i)} = \frac{h_M(j_i)}{h_B(j_i)r(h_S, j_i)} \geq \frac{(r(h_M, j_i))^{i+1}}{r(h_S, j_i)} \geq (r(h_S, j_i))^i.$$

Hence the pair (h_S, h_B) allows splitting above $|\sigma|$.

We now describe how we build the partial recursive tree S. We start by letting S_0 be an h_S-bushy subtree of T above σ with leaves of length l_1 or less such that if D_0 is the set of leaves of S_0 of length strictly smaller than l_1, then D_0 is h_B-small above σ. Since the terminal nodes of T are contained in B, such a tree must exist. We declare the nodes in D_0 terminal and the leaves of S_0 that are of length l_1 to be the *children of* σ. Throughout the construction we will maintain the property that if $\tau \in S$ has children in S, then they are all of the same length and that length is a splitting level for the pair (h_M, h_B).

At a stage s of the construction, we will have built a finite approximation S_s of S, and accumulated a set D_s of nodes on S_s that we have declared terminal. D_s will always be h_B-small above σ.

Suppose that $\tau \in S_s$ has a set C_τ of children of length l_i and that they are leaves of S_s. If we have not already done so, we initiate a search for a subset C'_τ of C_τ such that $C_\tau \setminus C'_\tau$ is h_B-small above τ, and for each $\rho \in C'_\tau$, there is a A_ρ, h_S-bushy above ρ such that the collection $\{A_\rho : \rho \in C'_\tau\}$ is pairwise Γ-splitting.

If $\tau \notin B$ then this search must terminate. To see why this is the case note first that B is h_B-small above τ. Let $\rho_0, ..., \rho_k$ be the strings in $C_\tau \setminus B$. Since l_i is a splitting level for (h_M, h_B), $h_M(l_i)/h_B(l_i) \geq r(h_M, l_i)$. Moreover, $w_{h_M}(l_i) \geq w_{h_S}(l_i) > k$. By Claim 5.9, there are $A_0, ..., A_k$, with A_j $h_M/2^{3+3k}$-big above ρ_j, that are pairwise Γ-splitting. Now

$$\frac{h_M(n)}{2^{3+3k}} \geq \frac{h_M(n)}{2^{3+3w_{h_S}(l_i)}} = \frac{h_M(n)}{r(h_S, l_i)} \geq h_S(n)$$

for $n \geq l_i$, so we can refine the A_j to subtrees that are h_S-bushy.

If C'_τ is found, then we extend each $\rho \in C'_\tau$ by A_ρ. Note that by Lemma 5.10, $D_s \cup (C_\tau \setminus C'_\tau)$ is h_B-small above σ, since D_s initially contains no extension of τ and $C_\tau \setminus C'_\tau$ is h_B-small above τ. So we can add $C_\tau \setminus C'_\tau$ to D_s.

Next, for each $\rho \in C'_\tau$ we wish to extend the leaves of A_ρ h_S-bushily to the next splitting level for (h_M, h_B). Let L_ρ be the set of leaves of A_ρ, and let $m = \max\{|\nu| : \nu \in L_\rho\}$. Let l be least splitting level for (h_M, h_B) greater equal to m. We begin a search for an $L'_\rho \subseteq L_\rho$ such that $L_\rho \setminus L'_\rho$ is h_B-small above ρ and above each $\nu \in L'_\rho$ there is an h_S-bushy tree with leaves of length l. Note that if $\rho \notin B$, this search must terminate. When we find such an L'_ρ, we extend all its elements h_S-bushily to level l, declaring the new leaves to be the children of ρ and add $L_\rho \setminus L'_\rho$ to D_s. The same argument as before shows that D_s remains h_B-small above σ.

The resulting tree S is h_S-bushy and if we let $D = \cup_s D_s$, then the new bad set $D \cup B$ is $2h_B$-small above σ. It is clear that the construction halts above a node $\tau \in S$ if it is either in B or we have declared it to be terminal by adding it to D, and so $B \cup D$ contains all the terminal nodes of S. By extending σ, we can ensure that $(h_S, 2h_B)$ allows splitting above $|\sigma|$. For such a σ, the condition $(\sigma, S, D \cup B, h_S, 2h_B)$ extends (σ, T, B, h_T, h_B) and forces $\Gamma^{f_\mathcal{G}} \geq_T f_\mathcal{G}$.

6. Appendix

Theorem 6.1 (Miller)**.** *Every function that is* DNC *relative to* $0'$ *is of hyperimmune degree.*

Proof. The argument is a modification of the proof by D. A. Martin of the fact that almost every real is of hyperimmune degree, as presented in Downey and Hirschfeldt [6]. We construct a reduction Ψ on ω^ω such that whenever Ψ^g is total for a $g \in \omega^\omega$, Ψ^g is not computably dominated. Moreover, any function that is DNC relative to $0'$ can compute such a g.

Construction of Ψ. The construction is comprised of subconstructions that operate simultaneously and independently above each string. If $\tau \in \omega^{<\omega}$ is of length n, then the subconstruction above τ attempts to do the following for each extension τi *of* τ *in succession*:

(1) Reserve a number m for τi such that $\Psi^{\tau i}(m)$ has not already been defined and m has not already been reserved for any initial segment or extension of τi.

(2) If $\varphi_n(m)$ converges, set $\Psi^{\tau i}(m) = \varphi_n(m) + 1$ and move on to $\tau(i+1)$. If $\varphi_n(m)$ never converges, then we say that the subconstruction above τ *halts at τi*.

In addition, at each stage s of the construction, let T_s be the (finite) set of strings for which s has been reserved. For each $f \in \omega^\omega \setminus [T_s]^\prec$, set $\Psi^f(s) = 0$.

Verification. Note that if g is such that no subconstruction halts at one of its initial segments, then Ψ^g is total: If m was reserved for an initial segment of g, then $\Psi^g(m)$ converges. Otherwise, at stage m we would have set $\Psi^g(m) = 0$.

Claim 6.2. *If Ψ^g is total, then it is not computably dominated.*

Proof. If $\tau \prec \tau i \prec g$ where τ is of length n, then the subconstruction above τ did not halt at τi. If it halted at τj for some $j < i$, then φ_n never converged on the number reserved for τj. If it proceeded beyond τi, then $\Psi^g(m) = \varphi_n(m) + 1$, where m is the number reserved for τi. In either case, φ_n does not dominate Ψ^g. $\square$

Claim 6.3. *If f is* DNC *relative to* $0'$ *then it computes a function g such that no subconstruction halts at an initial segment of g.*

Proof. We construct g by initial segments $\langle \tau_s \rangle_{s \in \omega}$. Given τ_s, let n be such that $\varphi_n^{0'}(n)$ converges to i if and only if the subconstruction above τ_s halts at $\tau_s j$. Then let $\tau_{s+1} = \tau_s f(n)$. Finally, let $g = \bigcup_s \tau_s$. $\square$

This concludes the proof of the theorem. $\square$

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